LE MATEMATICHE
Vol. LI (1996) – Fasc. II, pp. 335–347

GRAND SOBOLEV SPACES AND THEIR APPLICATIONS TO VARIATIONAL PROBLEMS

CARLO SBORDONE

Dedicated to Professor Francesco Guglielmino on his 70th birthday

For $q > 1$, Ω a bounded open set in $\mathbb{R}^n$, the grand Sobolev space $W_o^{1,q)}(\Omega)$ consists of all functions $u \in \bigcap_{0<\varepsilon\leq q-1} W_0^{1,q-\varepsilon}(\Omega)$ such that

$$\|u\|_{W_0^{1,q)}} = \sup_{0<\varepsilon\leq q-1} \left[\frac{\varepsilon}{|\Omega|} \int_\Omega |\nabla u|^{q-\varepsilon}\, dx \right]^{\frac{1}{q-\varepsilon}} < \infty. \tag{1.1}$$

This space, slightly larger than $W_0^{1,q}(\Omega)$, was introduced in [16] in connection with regularity properties of the Jacobians.
For $q = n$ in [9] imbedding theorems of Sobolev type were proved for functions $u \in W_0^{1,n)}(\Omega)$.
Here we report on recent use of grand Sobolev spaces to solve variational problems [14], [18].

Entrato in Redazione il 7 maggio 1997.

1. The grand Sobolev space $W^{1,q)}$.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. For $q > 1$ let us introduce a space slightly larger than $L^q(\Omega)$.
The grand L^q-space, denoted by $L^{q)} = L^{q)}(\Omega)$ consists of functions $h \in \bigcap_{0<\varepsilon\leq q-1} L^{q-\varepsilon}(\Omega)$ such that

$$\|h\|_{q)} = \sup_{0<\varepsilon\leq q-1} \left[\varepsilon \fint_\Omega |h|^{q-\varepsilon}\, dx \right]^{\frac{1}{q-\varepsilon}} < \infty \tag{1.1}$$

where $\fint_\Omega$ denotes the integral mean over Ω.
Note that $\|\cdot\|_{q)}$ is a norm and $L^{q)}(\Omega)$ is a Banach space.
This space was introduced in [16] in connection with regularity properties of the Jacobians, see also [11].
The Marcinkiewicz space $L^{q,\infty}(\Omega) =$ weak-$L^q(\Omega)$ and the Zygmund space $L^q \log^{-\alpha} L(\Omega)$, $\alpha \geq 0$, are defined according to the norms

$$\|h\|_{L^{q,\infty}} = \sup_{E\subset\Omega} |E|^{\frac{1}{q}} \fint_E |h|\, dx \tag{1.2}$$

$$\|h\|_{L^q \log^{-\alpha} L} = \inf \left\{ \lambda > 0 : \fint_\Omega \left|\frac{h}{\lambda}\right|^q \log^{-\alpha}\left(e + \left|\frac{h}{\lambda}\right| \right) dx < 1 \right\}. \tag{1.3}$$

The inclusions with grand L^q-space

$$L^q \subset L^{q,\infty} \subset L^{q)}$$

$$L^q \subset L^q \log^{-1} L \subset L^{q)} \subset \bigcap_{\alpha>1} L^q \log^{-\alpha} L$$

hold (see [11]).
L^∞ is not dense in $L^{q,\infty}$ nor in $L^{q)}$. In [2] the following formulas for the distance to L^∞ in these space were proved:

$$\operatorname{dist}_{L^{q,\infty}}(h, L^\infty) = \limsup_{|E|\to 0} |E|^{\frac{1}{q}} \fint_E |h|\, dx \tag{1.4}$$

$$\operatorname{dist}_{L^{q)}}(h, L^\infty) = \limsup_{\varepsilon\to 0} \left[\varepsilon \fint_\Omega |h|^{q-\varepsilon}\, dx \right]^{\frac{1}{q-\varepsilon}}. \tag{1.5}$$

We shall indicate by $L_0^{q)}$ the closure of L^∞.
The grand Sobolev space $W^{1,q)}(\Omega)$ consists of all functions

$$u \in \bigcap_{0<\varepsilon\le q-1} W^{1,q-\varepsilon}(\Omega)$$

such that $\nabla u \in L^{q)}(\Omega)$, equipped with the norm

$$\|u\|_{W^{1,q)}} = \|\nabla u\|_{L^{q)}} + \|u\|_{L^{q)}} .$$

We shall also consider the space $W_0^{1,q)}(\Omega)$ which consists of all functions u belonging to $\bigcap\limits_{0<\varepsilon\le q-1} W_0^{1,q-\varepsilon}(\Omega)$ such that the norm

$$\|u\|_{W_0^{1,q)}} = \|\nabla u\|_{L^{q)}}$$

is finite.
In the case $q = n$ an imbedding theorem of Sobolev-Trudinger type was established in [9] (see also [5],[6])

Theorem 1.1. *There exist* $c_1 = c_1(n)$, $c_2 = c_2(n)$ *such that for* $u \in W_0^{1,n)}(\Omega)$

$$\fint_\Omega \exp\left(\frac{|u|}{c_1|\Omega|^{\frac{1}{n}} \cdot \|u\|_{W_0^{1,n)}}}\right) dx \le c_2 ,$$

This means that $W_0^{1,n)}$ is imbedded in the Orlicz space EXP_α ($\alpha = 1$) which is defined according to the norm

$$\|f\|_{\mathrm{EXP}_\alpha} = \inf\left\{\lambda > 0 : \fint_\Omega \exp\left|\frac{f}{\lambda}\right|^\alpha dx \le 2\right\}.$$

It is well known that L^∞ is not dense into EXP_α. In [9] the following formulas for the distance

$$\begin{aligned}\mathrm{dist}_{\mathrm{EXP}_\alpha}(f, L^\infty) &= \inf\left\{\lambda > 0 : \fint_\Omega \exp\left|\frac{f}{\lambda}\right|^\alpha dx < \infty\right\} \\ &= e \cdot \limsup_{q\to\infty} \frac{1}{q}\left(\fint_\Omega |f|^{\alpha q}\, dx\right)^{\frac{1}{q}}\end{aligned}$$

were established. Moreover it is easy to check that, denoting with $\exp_\alpha$ the closure of L^∞ in EXP_α,

$$(\exp_\alpha)^* = L\log^{\frac{1}{\alpha}} L$$

$$(L\log^{\frac{1}{\alpha}} L)^* = \mathrm{EXP}_\alpha .$$

Let us compare Theorem 1.1 with well known imbedding for the Sobolev space $W_o^{1,n}$

$$W_0^{1,n} \subset VMO \tag{1.6}$$

where VMO is the class of functions $u \in L^1(\Omega)$ with vanishing mean oscillation, i.e.

$$\lim_{r\to 0} \fint_{B_r(x)} |u - u_{B_r}|\, dy = 0 \tag{1.7}$$

uniformly with respect to x, $B_r = B_r(x)$ the ball with radius r, centered at x, $u_{B_r} = \fint_{B_r} u$.
Formula (1.7) follows from Poincaré inequality and Jensen inequality:

$$\begin{aligned} \fint_{B_r} |u - u_{B_r}|\, dy &\le c \cdot r \fint_{B_r} |\nabla u|\, dy \le \\ &\le c \cdot r \left[\fint_{B_r} |\nabla u|^n\, dy\right]^{\frac{1}{n}} = c' \int_{B_r} |\nabla u|^n\, dy . \end{aligned} \tag{1.8}$$

If $u \in W_0^{1,1}$ and we assume only $|\nabla u| \in L^{n,\infty}$, then u belongs to BMO but not necessarily to VMO.

Taking into account (1.4), (1.8) in [2] it is proved that if $u \in W_0^{1,1}$ and $|Du| \in L_0^{n)}$ then $u \in \exp$, i.e.

$$\fint_\Omega \exp\left(\frac{|u|}{\lambda}\right) dx < \infty \qquad \text{for any } \lambda > 0.$$

Finally let us mention that in [7], [15] there are examples of functions $u \in W_0^{1,1}$ and such that $|\nabla u| \in L_0^{n)}$ and $u \notin BMO$.

2. Jacobian of $W^{1,n)}(\Omega, \mathbb{R}^n)$ mappings.

Let Ω be an open subset of $\mathbb{R}^n$ and let $f = (f^{(1)}, \dots, f^{(n)}) : \Omega \to \mathbb{R}^n$ be a mapping whose distributional differential $Df : \Omega \to \mathbb{R}^{n+n}$ is a locally integrable function on Ω with values in the space $\mathbb{R}^{n\times n}$ of all $n \times n$ matrices. The Jacobian determinant

$$J = J(x, f) = \det Df(x)$$

is point-wise defined a.e. in Ω.

When studying the integrability properties of the jacobian, the *natural* assumption for the integrability of Df is $|Df| \in L^n(\Omega)$, as it obviously implies that $J \in L^1(\Omega)$ from the Hadamard's inequality

$$|J(x, f)| \le |Df(x)|^n.$$

In case J is non negative (f an orientation preserving map), in [16] we have relaxed the natural assumptions on Df to ensure local integrability of the jacobian, proving that

$$|Df| \in L^{n)}(\Omega) \Rightarrow J \in L^1_{\mathrm{loc}}(\Omega).$$

The main steps for the proof are the following Proposition 2.1 without any assumption on the sign of J and its local versions in which J is assumed non negative. For $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let us indicate by Mh the Hardy-Littlewood maximal function

$$Mh(x) = \sup_{Q \ni x} ⨍_Q |h| dy$$

the supremum being taken over all subcubes of Ω containing the given point $x \in \Omega$.

Proposition 2.1. *Let* $-\infty < \varepsilon \le 1$ *and* $f \in W^{1,n-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n} (M|Df|)^{-\varepsilon} J(x, f)\, dx \le c(n)|\varepsilon| \int_{\mathbb{R}^n} |Df(x)|^{n-\varepsilon}\, dx\,. \tag{2.1}$$

A new proof of such estimate was recently given by L. Greco [12], relying on the following result of Acerbi-Fusco

Lemma 2.1. *For* $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, *and any* $t > 0$ *there exists* $g_t \in \mathrm{Lip}(\mathbb{R}^n)$ *such that* $g_t(x) = u(x)$ *for a.e.* $x \in \mathbb{R}^n$ *satisfying* $M|\nabla u|(x) \le t$ *and* $\|\nabla g_t\|_{L^\infty} \le c(n)t$.

Let us sketch the proof from [12].

Proof of (2.1). Fix $0 < \varepsilon < 1$, $f = (f^{(1)}, \dots, f^{(n)}) \in W^{1,n-\varepsilon}(\mathbb{R}^n, \mathbb{R}^n)$ and apply Lemma 2.1 with $u = f^{(1)}$, which gives $g = g_t \in W^{1,n-\varepsilon}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$. By Stokes theorem $\det D(g, f^{(2)}, \dots, f^{(n)})$ has zero integral over $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} \det D(g, f^{(2)}, \dots, f^{(n)})\, dx = 0$$

and so we can split for $t > 0$, as

$$\int_{M \le t} J\, dx = -\int_{M > t} \det DG\, dx \tag{2.2}$$

where $J = J(f, x)$, $G = (g, f^{(2)}, \dots, f^{(n)})$ and $M = M(x)$ stands for the maximal function of $|Df|$, that is here $M = M|Df|$.

Using the estimate on g given by Lemma 1, we easily deduce from (2.2)

$$\int_{M \le t} J\, dx \le c(n) t \int_{M > t} |Df|^{n-1}\, dx \tag{2.3}$$

for any $t > 0$.

Let us multiply both sides of (2.3) by $t^{-\varepsilon-1}$ and integrate over $(0, \infty)$ with respect to t; by Fubini we get

$$\int_{\mathbb{R}^n} M^{-\varepsilon} J\, dx \le c(n)\, \frac{\varepsilon}{1-\varepsilon} \int_{\mathbb{R}^n} M^{1-\varepsilon} |Df|^{n-1}\, dx\,. \tag{2.4}$$

The right hand side can be estimated by mean of Hölder inequality and the maximal theorem:

$$\int_{\mathbb{R}^n} (Mh)^{n-\varepsilon}\, dx \le c_1(n) \int_{\mathbb{R}^n} h^{n-\varepsilon}\, dx \qquad \varepsilon \text{ small}$$

as follows

$$\begin{aligned} \int_{\mathbb{R}^n} M^{1-\varepsilon} |Df|^{n-1}\, dx &\le \left(\int_{\mathbb{R}^n} M^{n-\varepsilon}\, dx \right)^{\frac{1-\varepsilon}{n-\varepsilon}} \left(\int_{\mathbb{R}^n} |Df|^{n-\varepsilon}\, dx \right)^{\frac{n-1}{n-\varepsilon}} \\ &\le c_2(n) \int_{\mathbb{R}^n} |Df|^{n-\varepsilon}\, dx\,. \end{aligned} \tag{2.5}$$

We immediately deduce (2.1) from (2.4) and (2.5).

Remark 1. By Young and Hadamard inequalities we deduce

$$|Df|^{-\varepsilon} J \leq (1-\varepsilon)(M|Df|)^{-\varepsilon} J + \varepsilon (M|Df|)^{1-\varepsilon} |Df|^{n-1}.$$

So by (2.3) and (2.4) we derive the sharper inequality

$$\int_{\mathbb{R}^n} |Df|^{-\varepsilon} J \, dx \leq c(n)\varepsilon \int_{\mathbb{R}^n} |Df|^{n-\varepsilon} \, dx \tag{2.6}$$

whose essence is the presence of factor ε in the right hand side.

Using (2.6) we deduce as in [16] the following

Theorem 2.1. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be an orientation preserving map such that $|Df| \in L^{n)}(\Omega)$, then $J \in L^1_{\mathrm{loc}}(\Omega)$ and*

$$\fint_B J \, dx \leq c(n) \|\nabla f\|_{L^{n)}(2B)}$$

for $B \subset 2B \subset \Omega$ concentric balls.

In [11] the following inequality for mappings $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (J(x,f)) \log |Df(x)| \, dx \leq c(n) \int_{\mathbb{R}^n} |Df(x)|^n \, dx \tag{2.7}$$

was proved.

We wish to give here a simple proof of (2.7) by mean of previous method.

Proposition 2.2. *For $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} (J(x,f)) \log MDf(x) \, dx \leq c(n) \int_{\mathbb{R}^n} |Df(x)|^n \, dx \, . \tag{2.8}$$

Proof. Like in the proof of Proposition 2.1, let us start with the identity

$$\int_{\mathbb{R}^n} \det D(g, f^{(2)}, \ldots, f^{(n)}) \, dx = 0$$

and split it for $t > 0$:

$$-\int_{M \leq t} J \, dx = \int_{M > t} \det DG \, dx \leq c(n) t \int_{M > t} |Df|^{n-1} \, dx$$

where $M = M|Df|$, $G = (g, f^{(2)}, \dots, f^{(n)})$. So we have

$$-\frac{1}{t}\int_{M\le t} J\,dx \le c(n)\int_{M>t} |Df|^{n-1}\,dx\,. \tag{2.9}$$

Since

$$\begin{aligned}\int_{\mathbb{R}^n} |J|\,dx \int_M^{\|Df\|_\infty} \frac{dt}{t} &\le \int_{\mathbb{R}^n} |Df|^{n-1}\,dx \int_M^{\|Df\|_\infty} \frac{|Df|}{M}\,dt\\ &\le \int_{\mathbb{R}^n} |Df|^{n-1}\,dx \cdot \|Df\|_\infty < \infty\end{aligned}$$

we can integrate (2.9) on $(0, \|Df\|_\infty)$ and apply Fubini, obtaining

$$\begin{aligned}-\int_0^{\|Df\|_\infty} \frac{1}{t}\,dt \int_{M\le t} J\,dx &= -\int_{M\le\|Df\|_\infty} J\,dx \int_M^{\|Df\|_\infty} \frac{dt}{t} =\\ &= \int_{\mathbb{R}^n} J(\log M - \log\|Df\|_\infty)\,dx = \int_{\mathbb{R}^n} J\log M\,dx\,.\end{aligned} \tag{2.10}$$

On the other hand

$$\begin{aligned}\int_0^{\|Df\|_\infty} dt \int_{M>t} |Df|^{n-1}\,dx &= \int_{\mathbb{R}^n} |Df|^{n-1}\,dx \int_0^{M\wedge\|Df\|_\infty} dt =\\ &= \int_{\mathbb{R}^n} |Df|^{n-1} M\,dx \le c'(n)\int_{\mathbb{R}^n} |Df|^n\,dx\end{aligned} \tag{2.11}$$

by Hölder inequality and the maximal theorem.
Inequalities (2.10) and (2.11) imply (2.8).
To obtain (2.7) we decompose the left hand side

$$\int_{\mathbb{R}^n} J\log|Df|\,dx = \int_{\mathbb{R}^n} J\log M|Df|\,dx + \int_{\mathbb{R}^n} J\log\frac{|Df|}{M|Df|}\,dx$$

and observe that the last integral is obviously convergent.

Note that we are not using that J belongs to the Hardy space $\mathcal{H}^1$ as it is proved by [4], nor that $\log M|Df|$ belongs to BMO as it is proved by [3].

3. Elliptic equations with right hand side in divergence form.

Let us consider the operator

$$Lu = \operatorname{div} \mathcal{A}(x, \nabla u) \tag{3.1}$$

in a regular domain $\Omega \subset \mathbb{R}^n$, where the mapping $\mathcal{A} = \mathcal{A}(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ verifies the "almost linear" conditions:

$$|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \le m|\xi - \eta| \tag{3.2}$$

$$m^{-1}|\xi - \eta|^2 \le \langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \tag{3.3}$$

$$\mathcal{A}(x, 0) = 0\,. \tag{3.4}$$

The natural setting for the Dirichlet problem

$$\begin{cases} Lu = \operatorname{div} F & \text{in } \Omega \\ u_{|\partial\Omega = 0} \end{cases} \tag{3.5}$$

corresponds to the assumption $F \in L^2(\Omega, \mathbb{R}^n)$. In this case classical results on monotone operators (see [19] e.g.) imply that there exists exactly one solution

$$u \in W_0^{1,2}(\Omega)$$

to problem (3.5), i.e.

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla \varphi \rangle = \int_\Omega \langle F, \nabla \varphi \rangle \tag{3.6}$$

for any $\varphi \in C_0^1(\Omega)$. Of course, by an approximation argument (3.6) extends to all $\varphi \in W_0^{1,2}$ as well.
Moreover if $F, G \in L^2(\Omega, \mathbb{R}^n)$ are given and $u, v \in W_0^{1,2}$ solve respectively

$$Lu = \operatorname{div} F \qquad Lv = \operatorname{div} G$$

then

$$\|\nabla u - \nabla v\|_{L^2} \le m\, \|F - G\|_{L^2}\,.$$

Using Meyers type results below the natural exponent it is possible to prove the following existence and uniqueness theorem for the Dirichlet problem (3.5) when F belongs to the grand L^2 space $L^{2)}(\Omega, \mathbb{R}^n)$.

Theorem 3.1. *There exists $c = c(m, n)$ such that, if $F, G \in L^{2)}(\Omega, \mathbb{R}^n)$ each of the equations*

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} F \tag{3.7}$$

$$\operatorname{div} \mathcal{A}(x, \nabla v) = \operatorname{div} G \tag{3.8}$$

has unique solution in the grand Sobolev space $W_0^{1,2)}$ and

$$\|\nabla u - \nabla v\|_{L^{2)}} \leq c\|F - G\|_{L^{2)}} . \tag{3.9}$$

Proof. From [9] we deduce that there exists $\varepsilon_0 = \varepsilon_0(m, n)$ such that if $F, G \in L^{2-\varepsilon}$, each of the equations (3.7), (3.8) has unique solution in $W_0^{1,2-\varepsilon}$ and

$$\int_\Omega |\nabla u - \nabla v|^{2-\varepsilon}\, dx \leq c(m, n) \int_\Omega |F - G|^{2-\varepsilon}\, dx \tag{3.10}$$

Multiplying by $0 < \varepsilon < \varepsilon_0$ and taking supremum over ε we immediately get inequality (3.9).

Remark. Many other inequalities can be deduced by (3.10), multiplying by functions $\rho = \rho(\varepsilon)$ and averaging with respect to ε (see [20]). For example if

$$F \in L^2(\log^{-a} L)(\log\log L)^{-b}$$

$(a > 0, b \geq 0)$ that is if

$$\int_\Omega |f|^2 \log^{-a}(e + |f|)[\log\log(2e + |f|)]^{-b}\, dx < \infty,$$

then problem (3.5) has a unique solution $u \in W_0^{1,1}(\Omega)$ such that

$$|Du| \in L^2(\log L)^{-a}(\log\log L)^{-b}$$

with a corresponding norm estimate.

4. Elliptic equations with a measure on the right hand side.

Let us consider the equation

$$\operatorname{div} F = \mu \tag{4.1}$$

where μ is a given Radon measure with finite mass on $\Omega \subset \mathbb{R}^n$.
We have the following result [8], [14] which naturally involves grand $L^{q)}$ spaces.

Lemma 4.1. *There exists $F \in L^{\frac{n}{n-1})}(\Omega, \mathbb{R}^n)$ such that* (4.1) *holds and*

$$\|F\|_{L^{\frac{n}{n-1})}(\Omega)} \le c(n) \int_\Omega |\,d\mu| \tag{4.2}$$

Proof. A solution to (4.1) can be expressed by the vectorial Riesz potential

$$F(x) = \frac{1}{n\omega_n} \int_\Omega \frac{x-y}{|x-y|^n}\, d\mu(y)$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$ [10].
If $1 \le s < \frac{n}{n-1}$ we can use the integral Minkowski inequality to obtain

$$\|F\|_s \le \frac{1}{n\omega_n} \int_\Omega \left\| \frac{1}{|x-y|^{n-1}} \right\|_s d\mu(y) \le \frac{1}{n\omega_n} \sup_{y\in\Omega} \left\| \frac{1}{|x-y|^{n-1}} \right\|_s \int_\Omega |d\mu|\,.$$

Since

$$\sup_{y\in\Omega} \left\| \frac{1}{|x-y|^{n-1}} \right\|_s^s = \left(\frac{\dfrac{n\omega_n}{n-1}}{\dfrac{n}{n-1-s}} \right) |\Omega|^{n-ns+s}$$

we immediately get (4.2) by taking the supremum over $s < \frac{n}{n-1}$.

Remark 1. If $d\mu = f(x)\,dx$ with $f \in L^1(\Omega)$ by an approximation argument we find that actually F belongs to $L_0^{\frac{n}{n-1})}(\Omega)$, that is

$$\lim_{\varepsilon\to 0} \varepsilon \int_\Omega |F|^{n-\varepsilon}\, dx = 0\,.$$

Let us see how the preceding lemma enables us to solve the question of existence and uniqueness of the solution of the Dirichlet problem

$$\begin{cases} Lu = \mu & \text{in } \Omega \\ u_{|\partial\Omega=0} \end{cases} \tag{4.3}$$

when L is the operator (3.1) under assumptions (3.2),(3.3),(3.4), in the particular case $n = 2$.

Theorem 4.1. *Let Ω be a regular domain in $\mathbb{R}^2$, μ a Radon measure with finite mass. Then there exists a unique solution $u \in W_0^{1,2)}$ to the Dirichlet problem* (4.3) *and*

$$\|u\|_{W_0^{1,2)}(\Omega)} \leq c(n) \int_\Omega |d\mu| .$$

Proof. We know that there exists $F \in L^{2)}$ such that $\operatorname{div} F = \mu$ and (4.2) holds for $n = 2$, from Lemma 4.1. By Theorem 3.1 we know that there exists $u \in W_o^{1,2)}$ such that

$$Lu = \operatorname{div} F = \mu$$

and

$$\|\nabla u\|_{L^{2)}} \leq c_1(n)\|F\|_{L^{2)}} .$$

The uniqueness follows from standard methods.

Remark 2. Much more general results can be proved in the case $n \geq 2$ (see [14]).

REFERENCES

[1] E. Acerbi - N. Fusco, *Semicontinuity problems in the calculus of variations,* Arch. Rat. Mech. Anal., 86 (1986), pp. 125-145.

[2] M. Carozza - C. Sbordone, *The distance to L^∞ in some function spaces and applications,* Diff. Integr. Equations (to appear).

[3] R.R. Coifman - R. Rochberg, *Another characterization of BMO,* Proc. AMS, 79-2 (1980), pp. 249-254.

[4] R.R. Coifman - P.L. Lions - Y. Meyer - S. Semmes, *Compensated compactness and Hardy spaces,* J. Math. Pures Appl., 72 (1993), pp. 247-286.

[5] D.E. Edmunds - P. Gurka - B. Opic, *On embeddings of logarithmic Bessel Potential spaces,* (to appear).

[6] D.E. Edmunds - M. Krbec, *Two limiting cases of Sobolev imbeddings,* Houston J. Math., 21 (1995), pp. 119-128.

[7] A. Fiorenza, *A summability condition ensuring BMO,* (to appear).

[8] A. Fiorenza - C. Sbordone, *Existence and uniqueness results for solutions of nonlinear equations with right hand side in* L^1, Studia Math. (to appear).

[9] N. Fusco - P.L. Lions - C. Sbordone, *Sobolev imbeddings theorems in borderline cases,* Proc. A. M. S., 124-2 (1996), pp. 561-565.

[10] D. Gilbarg - N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer, 1983.

[11] L. Greco, *A remark on the equality* $det Df = Det Df$, Diff. Integr. Equations, 6 (1993), pp. 1089-1100.

[12] L. Greco, *Sharp results on integrability of the Jacobian,* (to appear).

[13] L. Greco - T. Iwaniec, *New inequalities for the Jacobian,* Ann. Inst. Poincar, 11-1 (1994), pp. 17-35.

[14] L. Greco - T. Iwaniec - C. Sbordone, *Inverting the p-Harmonic Operator,* Manuscripta Math., 2 (1997), pp. 249-258.

[15] L. Greco - T. Iwaniec - C. Sbordone - B. Stroffolini, *Degree formulas for maps with nonintegrable Jacobian,* Top. Meth. Nonlinear Anal., 6 (1995), pp. 81-95.

[16] T. Iwaniec - C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses,* Arch. Rat. Mech. Anal., 119 (1992), pp. 129-143.

[17] T. Iwaniec - C. Sbordone, *Weak minima of variational integrals,* J. Reine Angew. Math, 454(1994), pp. 143-161.

[18] T. Iwaniec - C. Sbordone, *Riesz Transforms and elliptic pde's with VMO-coefficients,* (to appear).

[19] J. Necas, *Introduction to the theory of Nonlinear Elliptic Equations,* J. Wiley & Sons, 1986.

[20] C. Sbordone, *Nonlinear elliptic equations with right hand side in nonstandard spaces,* Rend. Sem. Mat. e Fis. Modena (to appear).

Dipartimento di Matematica e Applicazioni "R. Caccioppoli",
Università di Napoli "Federico II",
Via Cintia,
Complesso Monte S. Angelo,
80126 Napoli (ITALY)